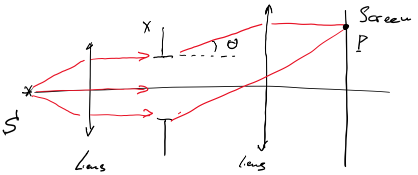
Fraunhofer diffraction as Fourier transformation

The scheme is very simple:

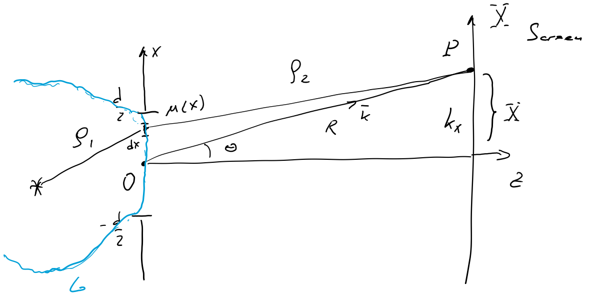


The screen has to be in far zone, i.e. very far away. If laboratory dimensions do not allow, then we can use second lens.

Now, let's calculate the diffraction pattern. We will perform this calculation at the whole screen, not just at central point as for Fresnel diffraction.

We'll take simple model of slit:

1) one dimensional
2) slit material does not influence field inside the slit.



Let's use H-F principle. For this we need to put around our source S with surface σ. In slit opening it will be plane surface. In other places the surface shape does not matter.

$$E_p = \int_\sigma K(\varphi) \frac{E_0'}{\rho_1 \rho_2} e^{ik(\rho_1+\rho_2)}\, d\sigma_n$$

$d\sigma_n$ is projection of $d\sigma$ perpendicular to ρ_1.

We will put x coordinate along the slit and

X coordinates on screen.

Our goal is to find amplitude in point P.

$$E_P = \int_{\sigma} K(\varphi) \frac{E_0'}{\rho_1 \rho_2} e^{ik(\rho_1+\rho_2)} \, d\sigma_n = \int_{\sigma} E(M) \, \frac{e^{ik\rho_2}}{\rho_2} \, d\sigma$$

This big integral we can simplify.

$E(M)$ is the field that will form in point M.

It will contain:

1) Property of the source E_0'
2) It will depend on ρ_1
3) It will contain tilt to ρ_1
4) $K(\varphi) = \text{const}$ as portion of first Fresnel zone is opened

Substituting source by radiating surface

To solve this integral, let's find ρ_2. We can find it from triangle OMP.

$$\rho_2^2 = R^2 + x^2 - 2Rx\sin\theta$$

$$R\sin\theta = X$$

$$\rho_2^2 = R^2 + x^2 - 2xX$$

Let's look at actual numbers:

$$\rho_2^2 = R^2\left(1 - \frac{2xX - x^2}{R^2}\right)$$

$$\begin{cases} x \sim 100\,\mu m \\ X \sim 1\,cm \\ R \sim 1\,m \end{cases} \Rightarrow \begin{cases} \dfrac{x^2}{R^2} \sim 10^{-8} \\ \dfrac{xX}{R^2} \sim 10^{-6} \end{cases} \ll 1$$

Hence, we can use

$$(1+x)^n \approx 1 + nx$$

$$n = \frac{1}{2}$$

$$\rho_2 = R\left(1 + \frac{2xX - x^2}{2R^2}\right) = R - \frac{xX}{R} + \frac{x^2}{2R^2}$$

$\searrow_0$

$\frac{x^2}{2R^2} \to 0$ is exactly the transition to Fraunhofer diffraction

$$\rho_2 = R - x\,\frac{X}{R} = R - x\sin\theta$$

Let's do couple of more mathematical manipulations.

$$\frac{e^{ik\rho_2}}{\rho_2} = \frac{e^{ikR}\,e^{-ikx\sin\theta}}{R - \underbrace{x\sin\theta}_{\searrow 0}} \qquad k \sim 10^7$$

$\searrow$ Strongly oscillating function

$x\sin\theta \ll R$

Finally,

$$E_p = \frac{e^{ikR}}{R}\int_{\sigma} E(\sigma)\,e^{-ikx\sin\theta}\,d\sigma$$

Only secondary sources within the slit are considered!

Since $k_x = k\sin\theta \Rightarrow E_p \sim \int_{-\frac{d}{2}}^{\frac{d}{2}} E(x)\,e^{-ik_x x}\,dx$

$E(x)$ - field inside the slit

We can play trick here:

$$E(x) = \begin{cases} E(x) & -\frac{d}{2} \le x \le \frac{d}{2} \\ 0, & |x| > \frac{d}{2} \end{cases}$$

$$E_p \sim \underbrace{\int_{-\infty}^{+\infty} E(x)\,e^{-ik_x x}\,dx}_{\text{Fourier image of } E(k_x)} = E(k_x)$$

Let's remember intermediate conclusion:

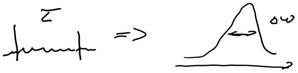
Amplitude of plane wave in direction θ $E_p \sim E(k_x)$

Let's recall FT:

$$\begin{cases} E(\omega) = \int_{-\infty}^{+\infty} E(t)\,e^{-i\omega t}\,dt \\ E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(\omega)\,e^{i\omega t}\,d\omega \end{cases} \qquad \begin{aligned} k_x &\to \omega \\ x &\to t \end{aligned}$$

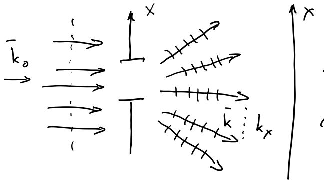
$$\begin{cases} E(k_x) = \int_{-\infty}^{+\infty} E(x)\, e^{-ik_x x}\, dx \\ E(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(k_x)\, e^{ik_x x}\, dk_x \end{cases}$$

Limitation in time



Limitation in space

k_x is spatial frequency.



When space limits plane wave it breaks into multiple waves (functions)

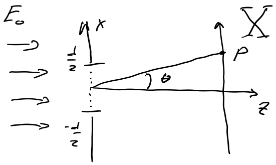
Demonstrations

Let's sum up:

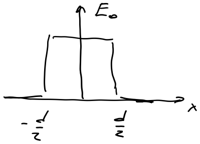
$$I_p \rightleftarrows E_p \rightleftarrows E(k_x) \rightleftarrows E(x)$$

In order to know intensity distribution on screen I_p we need to know field distribution within the slit.

Now, let's look at couple of examples.



$$E(x) = \begin{cases} E_0, & -\frac{d}{2} \le x \le \frac{d}{2} \\ 0, & |x| > \frac{d}{2} \end{cases}$$



Our algorithm

$$E(x) \to E(k_x) \to E_p \to I_p$$

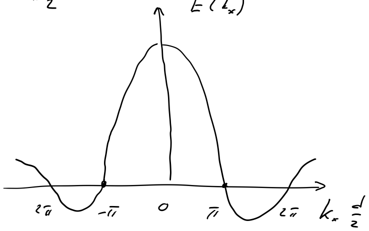
$$k_x = k \sin\theta$$

$$E(k_x) = \int_{-\infty}^{+\infty} E(x)\, e^{-ik_x x}\, dx =$$

$-\infty$

$$= E_0 \int_{-\frac{d}{2}}^{\frac{d}{2}} e^{-ik_x x}\,dx = \frac{E_0}{-ik_x}\left(e^{-ik_x\frac{d}{2}} - e^{ik_x\frac{d}{2}}\right) =$$

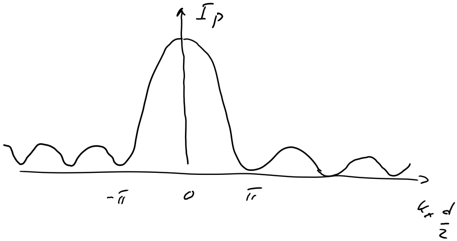
$$= d E_0 \frac{\sin k_x \frac{d}{2}}{k_x \frac{d}{2}}$$



To find amplitude:

$$I_p \sim |E(k_x)|^2$$

$$I_p = I_0 \frac{\sin^2 k_x \frac{d}{2}}{\left(\frac{k_x d}{2}\right)^2}$$

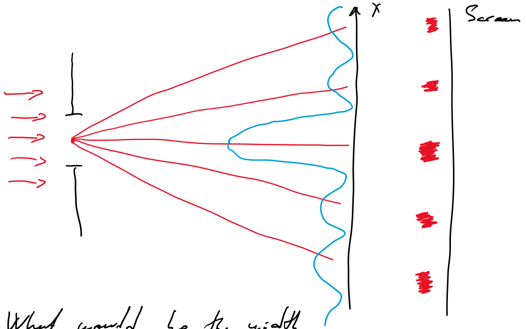


Minima will be observed at points, where

$$k_x \frac{d}{2} = m\pi \qquad \frac{2\pi}{\lambda}\sin\theta\,\frac{d}{2} = m\pi$$

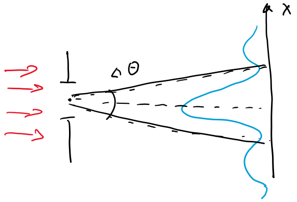
$$\boxed{d\sin\theta = m\lambda} \qquad m = \pm 1, \pm 2, \dots$$

What will we see on the screen?



What would be the width of central beam?

Intensity difference between beams?



$\Delta\theta \approx \frac{2\lambda}{d}$

uncertainty principle

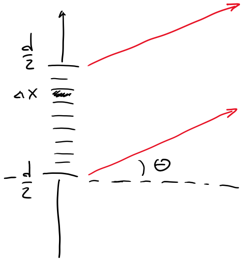
Please, derive

What does it mean?

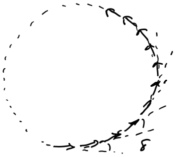
Thinner the slit, wider the central maximum.

Demonstration

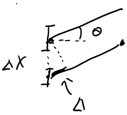
Diffraction on the slit we can discuss using graphical approach



Oscillations of single Δx we will represent as phasor



Please, note, amplitude from each Δx is the same as they all belong to the same Fresnel zone. However each of them will be delayed by δ.



$\delta = k \cdot \Delta = k \cdot \Delta x \sin\theta$

When the curve is connected, then we will have minimum

$$\text{min} \rightarrow \delta = \frac{2\pi m}{N} \qquad N = \frac{d}{\Delta x}$$

$m = 1 \rightarrow$ single turn

$m = 2 \rightarrow$ double turn

$$\frac{2\pi}{\lambda} \Delta x \sin\theta = \frac{2\pi m}{d} \Delta x$$

$$\Downarrow$$

$$\boxed{d \sin\theta = m\lambda}$$